Hermitian Forms & Hermitian Matrices

We want to look at the quadratic form:

$$p(\underline{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, \overline{x}_i \cdot x_j$$

Such a form is called <u>Hermitian</u> iff $p(\underline{x})$ is <u>real</u> for any vector $\underline{x}$ (the components of $\underline{x}$ can be arbitrary complex).

In a matrix notation:

$$p(\underline{x}) = \underline{x}^* \cdot A \cdot \underline{x}$$

conj. complex transpose.

<u>Question</u>: Which conditions exist on A to make $p(\underline{x})$ a Hermitian form?

$$p(\underline{x}) = \text{real} \iff p(\underline{x}) \equiv p^*(\underline{x})$$

$$p^*(\underline{x}) = [\underline{x}^* A \underline{x}]^* = \underline{x}^* A^* \underline{x}$$

$$\equiv p(\underline{x}) = \underline{x}^* A \underline{x}$$

for $\underline{\underline{\text{any}}}$ $\underline{x}$

$$\Rightarrow \quad \boxed{A^* = A}$$

- A matrix with this property is called a <u>Hermitian matrix</u>.

- An important subclass of the Hermitian matrices are the symmetric real matrices.

<u>Properties of Hermitian Matrices</u>:

(1) <u>Lemma</u>: The eigenvalues of Hermitian matrices are always real.

<u>Proof</u>: $A \cdot \underline{v}_i = \lambda_i \underline{v}_i$

$$\Rightarrow \underbrace{\underline{v}_i^* \cdot A \cdot \underline{v}_i}_{\text{real}} = \lambda_i \underbrace{\underline{v}_i^* \underline{v}_i}_{\text{pos. \& real}}$$

$$\Rightarrow \underline{\underline{\lambda_i = \text{real}}}$$

q.e.d.

(2) <u>Lemma</u>: Hermitian matrices have always a diagonal Jordan form $\Leftrightarrow$ The modal matrix is nonsingular.

<u>Proof</u>: Assume there exists a generalized eigenvector of grade $k > 1$

$$\Rightarrow \left| \begin{array}{l} (A - \lambda_i I)^k \cdot \underline{v}_i = \emptyset \\ (A - \lambda_i I)^{k-1} \cdot \underline{v}_i \neq \emptyset \end{array} \right|$$

$$\Rightarrow \left[\underbrace{(A-\lambda_i I)^k \underline{v}_i}_{\phi}\right]^* \cdot \left[(A-\lambda_i I)^{k-2} \cdot \underline{v}_i\right] = \phi$$

$$\Rightarrow \underline{v}_i^* \left[(A-\lambda_i I)^k\right]^* \cdot \left[(A-\lambda_i I)^{k-2} \cdot \underline{v}_i\right] = \phi$$

$$\Rightarrow \underline{v}_i^* \left[(A-\lambda_i I)^*\right]^k \cdot \left[(A-\lambda_i I)^{k-2} \cdot \underline{v}_i\right] = \phi$$

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$$\Rightarrow \underline{v}_i^* \cdot (A-\lambda_i I)^{k-1} \cdot (A-\lambda_i I)^{k-1} \cdot \underline{v}_i = \phi$$

$$\Rightarrow \underline{v}_i^* \cdot \left[(A-\lambda_i I)^*\right]^{k-1} \cdot \left[(A-\lambda_i I)^{k-1} \cdot \underline{v}_i\right] = \phi$$

$$\Rightarrow \underline{v}_i^* \cdot \left[(A-\lambda_i I)^{k-1}\right]^* \cdot \left[(A-\lambda_i I)^{k-1} \cdot \underline{v}_i\right] = \phi$$

$$\Rightarrow \left[(A-\lambda_i I)^{k-1} \cdot \underline{v}_i\right]^* \cdot \left[(A-\lambda_i I)^{k-1} \cdot \underline{v}_i\right] = \phi$$

$$\Rightarrow \left\| (A-\lambda_i I)^{k-1} \cdot \underline{v}_i \right\| = \phi$$

$$\Longrightarrow (A-\lambda_i I)^{k-1} \cdot \underline{v}_i = \phi$$

$\Rightarrow$ contradiction q.e.d.

(3) Lemma: The eigenvectors of a Hermitian matrix correspon=ding to different eigenvalues are orthogonal to each other.

Proof: Given $\lambda_i \neq \lambda_j \Rightarrow \underline{v}_i \neq \underline{v}_j$

$$A\underline{v}_i = \lambda_i \underline{v}_i \quad ; \quad A\underline{v}_j = \lambda_j \underline{v}_j$$

$$\Rightarrow \underline{v}_j^* A \underline{v}_i = \lambda_i \underline{v}_j^* \underline{v}_i \; ; \; \underline{v}_i^* A \underline{v}_j = \lambda_j \underline{v}_i^* \underline{v}_j$$

$$\Downarrow$$

$$\left[\underline{v}_j^* A \underline{v}_i\right]^* = \left[\lambda_i v_j^* \underline{v}_i\right]^*$$

$$\Downarrow$$

$$\Rightarrow \underline{v}_i^* A^* \underline{v}_j = \lambda_i \underline{v}_i^* \underline{v}_j$$

$$\Downarrow$$

$$\Rightarrow \underline{v}_i^* A \underline{v}_j = \lambda_i \underline{v}_i^* \underline{v}_j \equiv \lambda_j \underline{v}_i^* \underline{v}_j$$

$$\Rightarrow \underbrace{(\lambda_i - \lambda_j)}_{\neq \phi} \underbrace{\underline{v}_i^* \underline{v}_j}_{\Rightarrow = \phi} \equiv \phi$$

$$\Rightarrow \underline{v}_j \perp \underline{v}_i$$

q.e.d.

(4) Lemma: The modal matrix of a Hermitian matrix is a unitary matrix.

Proof: Follows directly from (3). If we have a multiple eigenvalue λ_i with multiplicity m_i $\Longrightarrow$

$$\text{Rank}\{(\lambda_i I - A)\} \equiv n - m_i$$

as there are no generalized eigenvectors.

$$\underline{\check{v}}_i \cdot \underline{\check{v}}_j = \phi \quad , \forall i \neq j$$

$$\Longrightarrow \quad (\lambda_i I - A)\underline{v}_i = \phi$$

spans a subspace in which it is possible to choose all vectors perpendicular to each other, and which as a whole is perpendicular to all other eigenvectors.

Lemma: The inverse of a unitary matrix is its Hermitian transpose.

Proof: $A = V \cdot \Lambda \cdot V^{-1}$

$$\equiv A^* = (V \cdot \Lambda \cdot V^{-1})^*$$

$$= (V^{-1})^* \cdot \Lambda^* \cdot V^*$$

diagonal & real

$$\equiv (V^*)^{-1} \cdot \Lambda \cdot (V^*)$$

$$\Rightarrow \boxed{V^{-1} \equiv V^*}$$

q.e.d.

Examples of Hermitian matrices:

(1) $M = A + A^*$; $A \in \mathbb{C}^{n\times n}$

e.g. $A = \begin{bmatrix} (a+jb) & (c+jd) \\ (e+jf) & (g+jh) \end{bmatrix}$

$$\Rightarrow M = A + A^* = \begin{bmatrix} (2a) & [(c+e)+j(d-f)] \\ [(c+e)+j(f-d)] & (2g) \end{bmatrix}$$

$\Rightarrow$ M is Hermitian as $M^* \equiv M$.

(2) $M = A^*A$; $A \in \mathbb{C}^{n\times m}$

e.g. $A = \begin{bmatrix} 2 & (5+j) & -7 \\ 0 & 3 & (2-3j) \end{bmatrix}$

$$\Rightarrow A^*A = \begin{bmatrix} 2 & 0 \\ (5-j) & 3 \\ -7 & (2+3j) \end{bmatrix} \cdot \begin{bmatrix} 2 & (5+j) & -7 \\ 0 & 3 & (2-3j) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & (10+2j) & -14 \\ (10-2j) & 35 & (-29-2j) \\ -14 & (-29+2j) & 62 \end{bmatrix}$$

$\Rightarrow A^*A$ is Hermitian.

(3) $M = A \cdot A^*$; $A \in \mathbb{C}^{n \times m}$

$$\begin{bmatrix} 2 & (5+j) & -7 \\ \phi & 3 & (2-3j) \end{bmatrix} \cdot \begin{bmatrix} 2 & \phi \\ (5-j) & 3 \\ -7 & (2+3j) \end{bmatrix} = \begin{bmatrix} 79 & (1-18j) \\ (1+18j) & 22 \end{bmatrix}$$

$\Rightarrow A \cdot A^*$ is Hermitian.

(4) $M = A - A^*$; $A \in \mathbb{C}^{n \times n}$

A as in (1)

$$\Rightarrow M = A - A^* = \begin{bmatrix} (2jb) & [(c-e)+j(d+f)] \\ [(e-c)+j(d+f)] & (2jh) \end{bmatrix}$$

$$= \begin{bmatrix} (2jb) & +[(c-e)+j(d+f)] \\ -[(c-e)-j(d+f)] & (2jh) \end{bmatrix}$$

$$\Rightarrow m_{ji} = -\overline{m_{ij}}$$

$\Rightarrow A - A^*$ is skew-Hermitian.

Lemma: Any matrix can be split into the sum of two matrices out of which one is Hermitian, the other is skew-Hermitian:

$$A = A_H + A_{SH}$$

Proof (by construction):

$$A_H = \frac{1}{2}(A + A^*) \quad \text{is Hermitian}$$
$$A_{SH} = \frac{1}{2}(A - A^*) \quad \text{is skew-Hermitian}$$

$$\overline{\qquad\qquad\qquad\qquad}$$

$$A_H + A_{SH} \equiv A \qquad \text{q.e.d.}$$

Lemma: A Hermitian matrix is called positive definite

$$(A > \emptyset)$$

pos. def.

iff the quadratic form:

$$\underline{x}^* A \underline{x} > \emptyset \quad ; \forall \underline{x} \neq (\underline{x} = \emptyset)$$

Example:

$M = A^* \cdot A$ is a Hermitian matrix. M is positive definite ($M > \emptyset$) since:

$$\underline{x}^* M \underline{x} = \underline{x}^* A^* A \underline{x} = (A\underline{x})^* \cdot (A\underline{x})$$
$$= \| A\underline{x} \|_2^2 > \emptyset \qquad \text{q.e.d.}$$

Of course: $\vec{M} = A \cdot A^*$ is also positive definite.

Lemma: The eigenvalues of a positive definite matrix are all positive and real.

Proof: $\quad A \cdot \underline{v}_i = \lambda_i \underline{v}_i$

$$\Rightarrow \underbrace{\underline{v}_i^* \cdot A \cdot \underline{v}_i}_{> \emptyset} = \lambda_i \cdot \underbrace{\underline{v}_i^* \cdot \underline{v}_i}_{> \emptyset}$$

$\Rightarrow \underline{\underline{\lambda_i > 0}}$ q.e.d.